Diagonalization of a coupled supersymmetric *t-J* model and its ground-state properties

2001 J. Phys. A: Math. Gen. 34 2787

(http://iopscience.iop.org/0305-4470/34/13/310)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 2787–2799 www.iop.org/Journals/ja PII: S0305-4470(01)18157-6

Diagonalization of a coupled supersymmetric t–J model and its ground-state properties

Sooa Park and K Lee

Department of Physics, Ewha Womans University, Seoul 120-750, South Korea

Received 17 October 2000

Abstract

We obtain an integrable two-leg supersymmetric t–J model through the algebraic Bethe ansatz scheme in the BFF grading. In this model, the two t–J chains interact with each other via a coupling constant κ. The model reduces to a two-chain Heisenberg spin model when the bosonic degrees of freedom are turned off and to two decoupled t–J chains when $\kappa = 0$. The construction of the diagonalized Hamiltonian yields the Bethe ansatz equations. We also obtain thermodynamic Bethe ansatz equations and calculate the magnetic susceptibility at a weak magnetic field and at zero temperature.

PACS numbers: 7127, 7510, 7540

1. Introduction

The spin-ladder system has been a candidate for describing high-T_c superconductivity phenomena since it appears in the realized form of Sr and other compounds, despite the fact that the maximum critical temperature has only reached 12 K as yet [1–5]. On the basis of many theoretical studies, it is well known that spin gaps appear in the even-legged Heisenberg spin-ladder model [6–8] and there have been further investigations including efforts to construct new integrable ladder systems [9–13]. However, despite the success of the model in giving pseudo-gap regime which is a general feature of a transition metal oxide with superconductivity, it has failed to be theoretically extended into the superconducting phase. This has led the attention of many physicists to a more realistic model which includes the charge degree of freedom so that one can consider the interaction between spinons and holons which is believed to play an essential role in the mechanism of high-T_c superconductivity [14]. Recently, the t–J ladder system has been studied as one of those models. Zhang and Rice first suggested the possibility that a t–J model could represent strongly correlated systems [15]. Although the Hamiltonian of this model has much simplified terms compared with the real systems, it is not easy for theorists to handle both exactly and perturbatively. On the other hand, the one-dimensional (1D) t–J model with supersymmetry, i.e. $J = 2t$, was solved exactly by a coordinate Bethe ansatz method [16,17] and also by an algebraic scheme [18]. In the latter, the model was treated with the three possible ways of FFB, BFF and FBF grading depending on which R-matrix was chosen.

0305-4470/01/132787+13$30.00

In this paper, we diagonalize a coupled supersymmetric t–J model with BBF grading and discuss ground-state properties of the model. In this grading the ground state is composed of only real rapidities, hence the calculation to obtain the ground-state properties is simpler than in other gradings and also the model is directly mapped on the single supersymmetric t–J model and the coupled Heisenberg model in special limits. The integrable Hamiltonian of the model is obtained in section 2 following a similar procedure in [9, 10] where an integrable family of two-chain Heisenberg spin-$\frac{1}{2}$ models has been discussed by extending the symmetry algebra. In section 3, the discrete Bethe ansatz equations are obtained by means of the algebraic diagonalization via the quantum inverse scattering method [21] for the BFF-graded version. These equations are reduced to the coupled Heisenberg spin-$\frac{1}{2}$ model when every lattice site has a spin-$\frac{1}{2}$ electron without vacancy. In section 4, thermodynamic Bethe ansatz equations are obtained using string hypothesis. Ground-state properties in a weak magnetic field are followed in section 5. Concluding remarks follow in section 6.

2. Model Hamiltonian

Let us start with the R-matrix of the 1D supersymmetric t–J model constructed by the algebraic method in [18],

$$R_{jl}(\lambda) = \frac{\mathrm{i}}{\lambda + \mathrm{i}} I_{jl} + \frac{\lambda}{\lambda + \mathrm{i}} P_{jl} \tag{2.1}$$

where λ is the spectral parameter and the matrices I_{jl} (P_{jl}) are the identity (permutation) operators acting on jth and lth spaces simultaneously. In this model, there are three possible situations at a site, that is, empty, occupied by spin-up, or by spin-down electron with $s = \frac{1}{2}$. We exclude double occupancies for the integrability. Since one can treat the empty sites as holes or bosons while the occupied ones are treated as fermions, the matrix elements of the identity and the permutation operator are given, respectively, by

$$\begin{aligned} I^{\alpha\beta}_{\gamma\sigma} &= \delta_{\alpha\beta}\delta_{\gamma\sigma} \\ P^{\alpha\beta}_{\gamma\sigma} &= \delta_{\alpha\sigma}\delta_{\gamma\beta}(-1)^{\epsilon_\alpha\epsilon_\gamma} \end{aligned} \tag{2.2}$$

where ϵ_α are the Grassmann parities with $\epsilon_\alpha = 0$ for the bosonic representations and $\epsilon_\alpha = 1$ for the fermionic ones. Hence they are also taken into account in defining the graded tensor product space, in which for any two linear operators A and B,

$$(A \otimes B)^{\alpha\beta}_{\gamma\sigma} = A_{\alpha\beta} B_{\gamma\sigma} (-1)^{\epsilon_\gamma(\epsilon_\alpha + \epsilon_\beta)}. \tag{2.3}$$

It is known that for any Yang's R-matrices acting on vector spaces in their irreducible representations, we can uniquely determine the extended R-matrices associated with tensor products of those representations [9, 22]. Hence, from equation (2.1), we can obtain a new R-matrix on the extended space,

$$\mathcal{R}(\lambda) = R_{j_2 j_3}(\lambda + \kappa) R_{j_2 j_4}(\lambda + \kappa - \kappa') R_{j_1 j_3}(\lambda) R_{j_1 j_4}(\lambda - \kappa') \tag{2.4}$$

with the shift parameters κ and κ' being coupling constants. For simplicity, we only consider the case of κ being equal to κ'. Defining an L operator on site n as $L_n(\lambda) = P_{mn} R_{mn}(\lambda)$, the monodromy matrix $\mathcal{T}(\lambda)$ is given by

$$\begin{aligned} &\mathcal{T}(\lambda) = \tau(\lambda + \kappa) \otimes \tau(\lambda) \\ &\tau(\lambda) = L_{2N_a}(\lambda) L_{2N_a - 1}(\lambda - \kappa) L_{2N_a - 2}(\lambda) L_{2N_a - 3}(\lambda - \kappa) \cdots L_2(\lambda) L_1(\lambda - \kappa) \end{aligned} \tag{2.5}$$

where N_a is the number of sites in a single chain. Note that the R-matrix satisfies the Yang–Baxter equation

$$\mathcal{R}(\lambda-\mu)\left(L_j(\lambda)\otimes L_j(\mu)\right)=\left(L_j(\mu)\otimes L_j(\lambda)\right)\mathcal{R}(\lambda-\mu). \tag{2.6}$$

After calculating the transfer matrix $T(\lambda)$ by taking a supertrace of the monodromy matrix as

$$T(\lambda)=\mathrm{str}[\mathcal{T}(\lambda)]=\mathcal{T}(\lambda)^{11}-\mathcal{T}(\lambda)^{22}-\mathcal{T}(\lambda)^{33} \tag{2.7}$$

the Hamiltonian of this system can be obtained by

$$H=-\frac{\mathrm{i}}{2}\left[\frac{\partial}{\partial\mu}\ln T(\mu)\right]_{\mu=0}-2\left(\frac{\kappa^2+2}{\kappa^2+1}\right)N_a. \tag{2.8}$$

The resulting Hamiltonian is written as

$$\begin{aligned}H=&-\frac{1}{\kappa^2+1}\sum_{j=1}^{N_a}\big[(X^{\sigma 0}_{2j-1}X^{0\sigma}_{2j}+X^{\sigma 0}_{2j}X^{0\sigma}_{2j+1}+\mathrm{h.c.})-X^{\sigma\tau}_{2j-1}X^{\tau\sigma}_{2j}-X^{\sigma\tau}_{2j}X^{\tau\sigma}_{2j+1}\\&+X^{00}_{2j-1}X^{00}_{2j}+X^{00}_{2j}X^{00}_{2j+1}+2I\big]\\&+\frac{\mathrm{i}\kappa}{2(\kappa^2+1)}\sum_{j=1}^{N_a}\big\{\big[X^{\sigma 0}_{2j-1}X^{0\tau}_{2j-2}(\delta_{\sigma\tau}X^{00}_{2j}+X^{\tau\sigma}_{2j})\\&+X^{\sigma 0}_{2j}X^{0\tau}_{2j-1}(\delta_{\sigma\tau}X^{00}_{2j-2}+X^{\tau\sigma}_{2j-2})+X^{\sigma 0}_{2j-2}X^{0\tau}_{2j}(\delta_{\sigma\tau}X^{00}_{2j-1}+X^{\tau\sigma}_{2j-1})\\&-X^{\sigma\tau}_{2j}X^{\tau\tau'}_{2j-1}X^{\tau'\sigma}_{2j-2}-\mathrm{h.c.}\big]\\&+\big[X^{\sigma 0}_{2j}X^{0\tau}_{2j+1}(\delta_{\sigma\tau}X^{00}_{2j-1}+X^{\tau\sigma}_{2j})+X^{\sigma 0}_{2j-1}X^{0\tau}_{2j}(\delta_{\sigma\tau}X^{00}_{2j+1}+X^{\tau\sigma}_{2j+1})\\&+X^{\sigma 0}_{2j+1}X^{0\tau}_{2j-1}(\delta_{\sigma\tau}X^{00}_{2j}+X^{\tau\sigma}_{2j})-X^{\sigma\tau}_{2j-1}X^{\tau\tau'}_{2j}X^{\tau'\sigma}_{2j+1}-\mathrm{h.c.}\big]\big\}\\&-\frac{\kappa^2}{2(\kappa^2+1)}\sum_{j=1}^{N_a}\big[(X^{\sigma 0}_{2j-2}X^{0\sigma}_{2j}+X^{\sigma 0}_{2j-1}X^{0\sigma}_{2j+1}+\mathrm{h.c.})-X^{\sigma\tau}_{2j-2}X^{\tau\sigma}_{2j}\\&-X^{\sigma\tau}_{2j-1}X^{\tau\sigma}_{2j+1}+X^{00}_{2j-2}X^{00}_{2j}+X^{00}_{2j-1}X^{00}_{2j+1}+2I\big].\end{aligned} \tag{2.9}$$

$X^{\alpha\beta}_j$ is the Hubbard operator of jth site with α and β being -1, 0 or 1, i.e.

$$\begin{aligned}X^{\sigma\tau}_j&=C^\dagger_{j\sigma}C_{j\tau}\qquad\sigma,\tau=\pm1\\X^{\sigma 0}_j&=(1-n_{j,-\sigma})C^\dagger_{j\sigma}\\X^{0\sigma}_j&=(1-n_{j,-\sigma})C_{j\sigma}\\X^{00}_j&=I-\sum_\sigma n_{j,\sigma}\end{aligned} \tag{2.10}$$

where $C^\dagger_{j\sigma}$ ($C_{j\sigma}$) is the creation (annihilation) operator of the spin-$\sigma/2$ electron of the jth site and the number operator is defined by $n_{j\sigma}=C^\dagger_{j\sigma}C_{j\sigma}$. If j is odd (even), then the electron lies in the first (second) chain. The first and third sums represent the nearest-neighbour interchain and intrachain interactions, respectively. The second sum, which shows a chiral interaction, is a consequence of the integrability of this model.

3. Discrete Bethe ansatz equations

As we mentioned above, there are three possible ways to describe our Hamiltonian by the algebraic Bethe ansatz method. Although all of them represent the same physics, the features of the resulting equations are somewhat different. From now on, we will focus on the 'BFF grading' along the lines of the extension of the two-chain Heisenberg spin model, where the system is considered to have a fermionic background. Hence we define the Grassmann parities as $\epsilon_1 = 0$ and $\epsilon_2 = \epsilon_3 = 1$, such that

$$P = \begin{pmatrix} e^{11} & e^{21} & e^{31} \\ e^{12} & -e^{22} & -e^{32} \\ e^{13} & -e^{23} & -e^{33} \end{pmatrix} \qquad (e^{ij})_{lm} = \delta_{il}\delta_{jm}. \tag{3.1}$$

Now, let $|\lambda_1\lambda_2\cdots\lambda_N\rangle$ be the eigenstate of the transfer matrix satisfying

$$\begin{aligned} T|\lambda_1\lambda_2\cdots\lambda_N\rangle &= t(\lambda+\kappa)t(\lambda)|\lambda_1\lambda_2\cdots\lambda_N\rangle \\ &= \varepsilon(\lambda+\kappa)\varepsilon(\lambda)|\lambda_1\lambda_2\cdots\lambda_N\rangle \end{aligned} \tag{3.2}$$

where ε is the eigenvalue of the single-chain transfer matrix t. The second line of equation (3.2) arises from the fact that if $t(\lambda)$ is diagonalized for the state, then T is also diagonalized. $t(\lambda)$ is obtained by taking a supertrace of the single chain monodromy matrix $\tau(\lambda)$ which reads

$$\tau(\lambda) = \begin{pmatrix} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ C_1 & C_2 & D \end{pmatrix}. \tag{3.3}$$

Using intertwining relations

$$\begin{aligned} &L(\lambda-\mu)^{a_1a_2}_{b_1b_2}\tau(\lambda)^{a_2a_3}_{\alpha_i\beta_i}\tau(\mu)^{b_2b_3}_{\beta_i\gamma_i}(-1)^{\epsilon_{a_1}\epsilon_{b_1}+\epsilon_{b_2}(\epsilon_{a_2}+\epsilon_{a_3})} \\ &\quad = \tau(\mu)^{b_1b_2}_{\alpha_i\beta_i}\tau(\lambda)^{a_1a_2}_{\beta_i\gamma_i}L(\lambda-\mu)^{a_2a_3}_{b_2b_3}(-1)^{\epsilon_{a_3}\epsilon_{b_3}+\epsilon_{a_1}(\epsilon_{b_1}+\epsilon_{b_2})} \end{aligned} \tag{3.4}$$

the commutation relations between the elements of τ are obtained as follows:

$$\begin{aligned} A_{ab}(\mu)C_c(\lambda) &= (-1)^{\epsilon_a\epsilon_p+\epsilon_a+\epsilon_b}\frac{r(\mu-\lambda)^{dc}_{pb}}{a(\mu-\lambda)}C_p(\lambda)A_{ad}(\lambda) \\ &\quad +(-1)^{(\epsilon_a+1)(\epsilon_b+1)}\frac{b(\mu-\lambda)}{a(\mu-\lambda)}C_b(\mu)A_{ac}(\lambda) \\ D(\mu)C_c(\lambda) &= \frac{1}{a(\mu-\lambda)}C_c(\lambda)D(\mu)+\frac{b(\lambda-\mu)}{a(\lambda-\mu)}C_c(\lambda)D(\mu) \\ C_{a_1}(\lambda_1)C_{a_2}(\lambda_2) &= r'(\lambda_2-\lambda_1)^{a_2b_1}_{a_1b_2}C_{b_2}(\lambda_2)C_{b_1}(\lambda_1) \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} r(\mu)^{ab}_{cd} &= b(\mu)I^{ab}_{cd}+a(\mu)P^{ab}_{cd} \\ r'(\mu)^{ab}_{cd} &= b(\mu)I^{ab}_{cd}+a(\mu)P'^{ab}_{cd} \qquad a,b,c,d=1,2 \\ P'^{ab}_{cd} &= \delta_{ad}\delta_{bc}(-1)^{(\epsilon_a+1)(\epsilon_c+1)} \\ a(\mu) &= \frac{\mu}{\mu+\mathrm{i}} \\ b(\mu) &= \frac{\mathrm{i}}{\mu+\mathrm{i}}. \end{aligned}$$

In terms of the elements of $\tau(\lambda)$, the eigenstate of the transfer matrix is written as

$$|\lambda_1 \cdots \lambda_N\rangle = \sum_{a_1 \cdots a_N} F^{a_N \cdots a_1} C_{a_1}(\lambda_1) \cdots C_{a_N}(\lambda_N)|0\rangle \tag{3.6}$$

where the vacuum state $|0\rangle$ means that all sites are empty. Using the relations in (3.5) and

$$\begin{aligned} D(\mu)|0\rangle &= \left[\frac{a(\mu)a(\mu-\kappa)}{a(-\mu)a(-\mu+\kappa)}\right]^{N_a}|0\rangle \\ A_{ab}(\mu)|0\rangle &= [a(\mu)a(\mu-\kappa)]^{N_a}|0\rangle\delta_{ab} \end{aligned} \tag{3.7}$$

we obtain the transfer matrix t acting on the state $|\lambda_1 \cdots \lambda_N\rangle$

$$\begin{aligned} t(\mu)|\lambda_1 \cdots \lambda_N\rangle &= [A_{11}(\mu) - A_{22}(\mu) - D(\mu)]|\lambda_1 \cdots \lambda_N\rangle \\ &= -\left[\frac{a(\mu)a(\mu-\kappa)}{a(-\mu)a(-\mu+\kappa)}\right]^{N_a}\left[\prod_{j=1}^{N}\frac{1}{a(\mu-\lambda_j)}\right]F^{a_N\cdots a_1}\prod_{l=1}^{N}C_{a_l}(\lambda_l)|0\rangle \\ &\quad +[a(\mu)a(\mu-\kappa)]^{N_a}\left[\prod_{j=1}^{N}\frac{1}{a(\mu-\lambda_j)}\right]t'(\mu)^{b_1\cdots b_N}_{a_1\cdots a_N}F^{a_N\cdots a_1}\prod_{l=1}^{N}C_{b_l}(\lambda_l)|0\rangle \\ &\quad +\sum_{k=1}^{N}\left[\Lambda(\lambda_k)^{p_1\cdots p_N}_{a_1\cdots a_N} - \tilde{\Lambda}(\lambda_k)^{p_1\cdots p_N}_{a_1\cdots a_N}\right]F^{a_N\cdots a_1}\,C_{p_k}(\mu)\prod_{l=1(\neq k)}^{N}C_{p_l}(\lambda_l)|0\rangle. \end{aligned} \tag{3.8}$$

Here $t'(\mu)$, $\Lambda(\lambda_k)$ and $\tilde{\Lambda}(\lambda_k)$ are the expressions as follows:

$$\begin{aligned} t'(\mu)^{b_1\cdots b_N}_{a_1\cdots a_N} &= L'(\mu-\lambda_N)^{bd_{N-1}}_{b_N a_N}L'(\mu-\lambda_{N-1})^{d_{N-1}d_{N-2}}_{b_{N-1}a_{N-1}}\cdots L'(\mu-\lambda_2)^{d_2 d_1}_{b_2 a_2}L'(\mu-\lambda_1)^{d_1 b}_{b_1 a_1} \\ &\quad \times(-1)^{\epsilon_b+\sum_{i=1}^{N-1}(\epsilon_b+\epsilon_{d_i})(\epsilon_{b_i}+1)} \\ \Lambda(\lambda_k)^{p_1\cdots p_N}_{a_1\cdots a_N} &= [a(\lambda_k)a(\lambda_k-\kappa)]^{N_a}\frac{b(\mu-\lambda_k)}{a(\mu-\lambda_k)}\left[\prod_{j=1(\neq k)}^{N}\frac{1}{a(\lambda_k-\lambda_j)}\right]\left[\prod_{j=1}^{k-1}\frac{a(\lambda_k-\lambda_j)}{a(\lambda_j-\lambda_k)}\delta_{p_j a_j}\right] \\ &\quad \times(-1)^{\epsilon_{p_k}\left[\sum_{j=1}^{k-1}(\epsilon_{a_j}+1)+\sum_{j=k+1}^{N}(\epsilon_{p_j}+1)\right]+\sum_{j=k+1}^{N-1}\epsilon_{d_j}+\sum_{j=k+1}^{N}\epsilon_{d_j}\epsilon_{p_j}+\sum_{j=1}^{k}(\epsilon_{a_j}+1)} \\ &\quad \times L'(\lambda_k-\lambda_N)^{p_k d_{N-1}}_{p_N a_N}L'(\lambda_k-\lambda_{N-1})^{d_{N-1}d_{N-2}}_{p_{N-1}a_{N-1}}\cdots L'(\lambda_k-\lambda_{k+1})^{d_{k+1}a_k}_{p_{k+1}a_{k+1}} \\ \tilde{\Lambda}(\lambda_k)^{b_1\cdots b_N}_{a_1\cdots a_N} &= \left[\frac{a(\lambda_k)a(\lambda_k-\kappa)}{a(-\lambda_k)a(-\lambda_k+\kappa)}\right]^{N_a}\frac{b(\lambda_k-\mu)}{a(\lambda_k-\mu)}\left[\prod_{j=1(\neq k)}^{N}\frac{1}{a(\lambda_k-\lambda_j)}\right] \\ &\quad \times\delta_{a_N b_N}\delta_{a_{N-1}b_{N-1}}\cdots\delta_{a_{k+1}b_{k+1}} \\ &\quad \times r'(\lambda_k-\lambda_{k-1})^{a_k b_{k-1}}_{a_{k-1}c_{k-1}}r'(\lambda_k-\lambda_{k-2})^{c_{k-1}b_{k-2}}_{a_{k-2}c_{k-2}}\cdots r'(\lambda_k-\lambda_2)^{c_3 b_2}_{a_2 c_2}r'(\lambda_k-\lambda_1)^{c_2 b_1}_{a_1 b_k} \end{aligned}$$

with

$$L'(\mu)^{ab}_{cd} = (-1)^{\epsilon_b\epsilon_d}r(\mu)^{ad}_{cb}.$$

It is straightforward from equation (3.8) to show that t is diagonalized if the following two conditions are satisfied at the same time:

$$\Lambda(\lambda_k)^{p_1\cdots p_N}_{a_1\cdots a_N}F^{a_N\cdots a_1} = \tilde{\Lambda}(\lambda_k)^{p_1\cdots p_N}_{a_1\cdots a_N}F^{a_N\cdots a_1} \tag{3.9}$$

$$t'(\mu)^{b_1\cdots b_N}_{a_1\cdots a_N}F^{a_N\cdots a_1} = \varepsilon'(\mu)F^{b_N\cdots b_1}. \tag{3.10}$$

Equation (3.9) makes the unwanted off-diagonal terms in (3.8) vanish and equation (3.10) causes the second term to be diagonal. After some tedious calculations, equation (3.9) reduces to

$$t'(\lambda_k)_{a_1\cdots a_N}^{f_1\cdots f_N} F^{a_N\cdots a_1} = [a(-\lambda_k)a(\kappa-\lambda_k)]^{-N_a} F^{f_N\cdots f_1} \tag{3.11}$$

where λ_k is not arbitrary contrary to the arbitrary μ in (3.10). Equation (3.10) is exactly the same as the nested Bethe ansatz obtained in the single-chain supersymmetric t–J model as expected. Hence, equation (3.10) is rewritten as the algebraic equation

$$t'(\mu)|\nu_1\cdots\nu_M\rangle = \varepsilon'(\mu)|\nu_1\cdots\nu_M\rangle \tag{3.12}$$

where ν_i are the rapidities in this nesting and M denotes the number of empty sites. This nested eigenequation yields

$$\varepsilon'(\mu) = \prod_{l=1}^{M}\frac{1}{a(\mu-\nu_l)}\left[\prod_{j=1}^{N}a(\mu-\lambda_j) - \prod_{j=1}^{N}\frac{a(\mu-\lambda_j)}{a(\lambda_j-\mu)}\right] \tag{3.13}$$

and

$$\prod_{j=1}^{N}a(\lambda_j-\nu_l) = 1 \qquad l=1,\ldots,M. \tag{3.14}$$

The latter equations come from the condition of vanishing of the unwanted terms. It is clear that from equations (3.11) and (3.13)

$$[a(-\lambda_k)a(\kappa-\lambda_k)]^{-N_a} = \left[\prod_{j=1(\neq k)}^{N}\frac{a(\lambda_k-\lambda_j)}{a(\lambda_j-\lambda_k)}\right]\prod_{l=1}^{M}\frac{1}{a(\lambda_k-\nu_l)}. \tag{3.15}$$

Hence the eigenvalue of $t(\mu)$ becomes

$$\begin{aligned}\varepsilon(\mu) = &-\left[\frac{a(\mu)a(\mu-\kappa)}{a(-\mu)a(-\mu+\kappa)}\right]^{N_a}\prod_{j=1}^{N}\frac{1}{a(\mu-\lambda_j)} \\ &+[a(\mu)a(\mu-\kappa)]^{N_a}\left[\prod_{j=1}^{N}\frac{1}{a(\mu-\lambda_j)}\right]\varepsilon'(\mu).\end{aligned} \tag{3.16}$$

Equations (3.14) and (3.15) bring forth the Bethe ansatz equations and the total energy of our model can be obtained by substituting equation (3.16) into

$$E = -\frac{\mathrm{i}}{2}\left[\frac{\partial}{\partial\mu}\ln\varepsilon(\mu+\kappa)\varepsilon(\mu)\right]_{\mu=0} - 2\left(\frac{\kappa^2+2}{\kappa^2+1}\right)N_a. \tag{3.17}$$

With the replacement of $\lambda_j \to \frac{1}{2}(\lambda_j+\kappa-\mathrm{i})$ and $\nu_l \to \frac{1}{2}(\nu_l+\kappa-2\mathrm{i})$, we obtain, the so-called discrete Bethe ansatz equations

$$\begin{aligned}&\left(\frac{\lambda_k+\kappa+\mathrm{i}}{\lambda_k+\kappa-\mathrm{i}}\right)^{N_a}\left(\frac{\lambda_k-\kappa+\mathrm{i}}{\lambda_k-\kappa-\mathrm{i}}\right)^{N_a} \\ &\qquad = -\left(\prod_{j=1}^{N}\frac{\lambda_k-\lambda_j+2\mathrm{i}}{\lambda_k-\lambda_j-2\mathrm{i}}\right)\left(\prod_{l=1}^{M}\frac{\lambda_k-\nu_l-\mathrm{i}}{\lambda_k-\nu_l+\mathrm{i}}\right) \qquad k=1,\ldots,N\end{aligned}$$

$$\prod_{j=1}^{N}\frac{\lambda_j-\nu_l-\mathrm{i}}{\lambda_j-\nu_l+\mathrm{i}} = 1 \qquad l=1,\ldots,M \tag{3.18}$$

with the energy

$$E = -2\sum_{j=1}^{N}\left[\frac{1}{(\lambda_j+\kappa)^2+1}+\frac{1}{(\lambda_j-\kappa)^2+1}\right]. \tag{3.19}$$

These equations reduce to those for the single-chain supersymmetric t–J model with $2N_a$-lattice sites [18] when the coupling constant κ is zero and those for the spin-$\frac{1}{2}$ Heisenberg ladder with two legs in the limit of no empty sites [9, 10, 19] as expected.

At zero temperature, all rapidities have real values giving the lowest energy. Hence, after taking the logarithm of (3.18) directly and introducing the usual particle (hole) distribution functions $\rho(\rho_h)$ for the rapidity λ and $\sigma(\sigma_h)$ for ν [23] in the thermodynamic limit where $N_a \to \infty$ but with N/N_a and M/N_a remaining finite, the discrete Bethe ansatz equations (3.18) are rewritten as

$$\begin{aligned}\rho(\lambda)+\rho_h(\lambda) &= \tfrac{1}{2}\left[a_1(\lambda+\kappa)+a_1(\lambda-\kappa)\right]-a_2*\rho(\lambda)+a_1*\sigma(\lambda)\\ \sigma(\lambda)+\sigma_h(\lambda) &= a_1*\rho(\lambda)\end{aligned} \tag{3.20}$$

and the energy in (3.19) is expressed as

$$\frac{E}{2N_a} = -2\pi\int\left[a_1(\lambda-\kappa)+a_1(\lambda+\kappa)\right]\rho(\lambda)\,\mathrm{d}\lambda. \tag{3.21}$$

Here, $a_n(\lambda)$ is defined as

$$\begin{aligned}a_n(\lambda) &= \frac{1}{\pi}\frac{n}{\lambda^2+n^2} \qquad & n>0\\ a_n(\lambda) &= 0 & n\leqslant 0\end{aligned}$$

and $*$ denotes the convolution defined by

$$f*g(\lambda) = \int \mathrm{d}\lambda'\, f(\lambda-\lambda')g(\lambda').$$

In the case where half of the sites are occupied by spin-up electrons and the rest by spin-down, i.e. $N = N_a$ and $M = 0$, the distribution functions satisfy

$$\begin{aligned}\rho(\lambda) &= \tfrac{1}{2}\left[g_0(\lambda+\kappa)+g_0(\lambda-\kappa)\right]\\ \sigma_h(\lambda) &= \tfrac{1}{2}\left[g_1(\lambda+\kappa)+g_1(\lambda-\kappa)\right]\end{aligned} \tag{3.22}$$

with

$$g_n(\lambda) \equiv \frac{1}{2\pi}\int \mathrm{d}\omega\,\frac{\mathrm{e}^{-n|\omega|}}{2\cosh|\omega|}\mathrm{e}^{-\mathrm{i}\lambda\omega}$$

since $\rho_h(\lambda) = \sigma(\lambda) = 0$. Inserting these results into the energy in (3.21), we obtain the ground-state energy per site as

$$\frac{E_0}{2N_a} = -2\pi\left[g_1(0)+g_1(2\kappa)\right]. \tag{3.23}$$

Note that this energy in (3.23) is the same as that of the two-leg Heisenberg spin ladder coupled by κ [10, 19] and reduces to that of the single Heisenberg spin chain, $-2\ln 2$, for $\kappa = 0$ [20], as expected.

4. Thermodynamic equations

When we turn on a magnetic field or temperature, it is necessary to consider the complex solutions of rapidities in the discrete Bethe ansatz equations for the lowest free energy [24]. Using the string hypothesis, $\lambda_k^{\pm} = \nu_k' \pm \mathrm{i}$, $k = 1, 2, \ldots, M'$ and $\lambda_n^{\alpha,k} = \lambda_n^{\alpha} + \mathrm{i}(n + 1 - 2k)$ for $k = 1, 2, \ldots, n$, $\alpha = 1, 2, \ldots, M_n$, and $n = 1, 2, \ldots, \infty$. Note that $\sum_n^{\infty} nM_n = N - 2M'$. The ν_k not coupled with $\lambda_k^{\pm}$ are all real. After replacing the rapidities in equation (3.18) with the above string solutions and taking logarithms of both sides, we have, in the thermodynamic limit,

$$\begin{aligned}
\rho_{nh}(\lambda) &= \tfrac{1}{2}\left[a_n(\lambda+\kappa) + a_n(\lambda-\kappa)\right] - \sum_{m=1}^{\infty} A_{nm} * \rho_m(\lambda) - (a_{n-2} + a_n + a_{n+2}) * \sigma'(\lambda) \\
&\quad + \delta_{n,1} a_1 * \sigma'(\lambda) + a_n * \sigma(\lambda) \\
\sigma'(\lambda) + \sigma_h'(\lambda) &= \tfrac{1}{2}\left[a_2(\lambda+\kappa) + a_2(\lambda-\kappa)\right] - \sum_{m=2}^{\infty} \left[a_{m-2} + a_m + a_{m+2}\right] * \rho_m(\lambda) \\
&\quad + a_1 * \rho_1(\lambda) - a_4 * \sigma'(\lambda) + a_2 * \sigma(\lambda) \\
\sigma(\lambda) + \sigma_h(\lambda) &= \sum_{m=1}^{\infty} a_m * \rho_m(\lambda) + a_2 * \sigma'(\lambda)
\end{aligned} \tag{4.1}$$

where $\rho_n(\rho_{nh})$, $\sigma'(\sigma_h')$ and $\sigma(\sigma_h)$ are the particle (hole) distribution functions for the rapidities λ_n^{α}, ν_k' and ν_k, respectively, and

$$A_{nm}(\lambda) = a_{|n-m|}(\lambda) + 2\left[a_{|n-m|+2}(\lambda) + \cdots + a_{n+m-2}(\lambda)\right] + a_{n+m}(\lambda) \qquad n \neq m$$

$$A_{nn}(\lambda) = \delta(\lambda) + 2\left[a_2(\lambda) + a_4(\lambda) + \cdots + a_{2n-2}(\lambda)\right] + a_{2n}(\lambda).$$

The energy in (3.19) is also rewritten as

$$\begin{aligned}
\frac{E}{2N_a} &= -2\pi \sum_{n=1}^{\infty} \int \mathrm{d}\lambda\, \rho_n(\lambda)\left[a_n(\lambda+\kappa) + a_n(\lambda-\kappa)\right] \\
&\quad -2\pi \int \mathrm{d}\lambda\, \sigma'(\lambda)\left[a_2(\lambda+\kappa) + a_2(\lambda-\kappa)\right].
\end{aligned} \tag{4.2}$$

Now we consider the temperature effect by adding the entropy contribution to the energy. Since the entropy is defined as the logarithm of the number of all possible states, the entropy per site can be written as

$$\begin{aligned}
\frac{S}{2N_a} &= \sum_{n=1}^{\infty} \int \mathrm{d}\lambda\left[(\rho_n + \rho_{nh})\ln(\rho_n + \rho_{nh}) - \rho_n \ln \rho_n - \rho_{nh} \ln \rho_{nh}\right] \\
&\quad + \int \mathrm{d}\lambda\left[(\sigma + \sigma_h)\ln(\sigma + \sigma_h) - \sigma \ln \sigma - \sigma_h \ln \sigma_h\right] \\
&\quad + \int \mathrm{d}\lambda\left[(\sigma' + \sigma_h')\ln(\sigma' + \sigma_h') - \sigma' \ln \sigma' - \sigma_h' \ln \sigma_h'\right].
\end{aligned} \tag{4.3}$$

With an external magnetic field H, the magnetization per site is expressed as

$$\frac{S_z}{2N_a} = \frac{1}{2}\left[1 - \sum_{n=1}^{\infty} 2n \int \mathrm{d}\lambda\, \rho_n(\lambda) - 3\int \mathrm{d}\lambda\, \sigma'(\lambda) - \int \mathrm{d}\lambda\, \sigma(\lambda)\right]. \tag{4.4}$$

Collecting equations (4.2)–(4.4), and introducing the Lagrange multiplier for the constraint that the total number of electrons, $2N_a - M$, remains constant, we can express the free energy

$G = E - TS - HS_z - A(2N_a - M)$ in terms of the distribution functions. The Lagrange multiplier A is the chemical potential of the system.

The minimization of our free energy over the distribution functions gives us the following thermodynamic Bethe ansatz equations:

$$\begin{aligned}
\ln(1+\rho_{nh}/\rho_n) &= \frac{1}{T}\left[nH - 2\pi\left(a_n(\lambda+\kappa)+a_n(\lambda-\kappa)\right)\right] + \sum_{m=1}^{\infty} A_{nm} * \ln(1+\rho_m/\rho_{mh}) \\
&\quad -a_n * \ln(1+\sigma/\sigma_h) + (a_{n-2}+a_n+a_{n+2}-\delta_{n,1}a_1) * \ln(1+\sigma'/\sigma'_h) \\
\ln(\sigma'_h/\sigma') &= \frac{1}{T}\left[\frac{3H}{2} + A - 2\pi\left(a_2(\lambda+\kappa)+a_2(\lambda-\kappa)\right)\right] - a_1 * \ln(1+\rho_1/\rho_{1,h}) \\
&\quad + \sum_{n=1}^{\infty}(a_{n-2}+a_n+a_{n+2}) * \ln(1+\rho_n/\rho_{nh}) - a_2 * \ln(1+\sigma/\sigma_h) \\
&\quad + a_4 * \ln(1+\sigma'/\sigma'_h) \\
\ln(\sigma_h/\sigma) &= \frac{1}{T}\left(-\frac{H}{2}+A\right) - \sum_{n=1}^{\infty} a_n * \ln(1+\rho_n/\rho_{nh}) - a_2 * \ln(1+\sigma'/\sigma'_h).
\end{aligned} \tag{4.5}$$

Then, using these equations the free energy per site is reduced to a simple form,

$$\begin{aligned}
\frac{G-E_0}{2N_a} &= -A - \frac{T}{2}\int \mathrm{d}\lambda\,[g_0(\lambda+\kappa)+g_0(\lambda-\kappa)]\ln(1+\rho_{1,h}/\rho_1) \\
&\quad -\frac{T}{2}\int \mathrm{d}\lambda\,[g_1(\lambda+\kappa)+g_1(\lambda-\kappa)]\left[\ln(1+\sigma/\sigma_h)+\ln(1+\sigma'/\sigma'_h)\right].
\end{aligned} \tag{4.6}$$

The next section is devoted to obtaining some special solutions of (4.5) and the magnetic susceptibility for a sufficiently weak magnetic field in the zero-temperature limit when κ is sufficiently small.

5. Zero-temperature limit

Let us introduce the usual energy potentials, ε_n, ϕ' and ϕ, defined as $\rho_{nh}/\rho_n = \mathrm{e}^{\varepsilon_n/T}$, $\sigma'_h/\sigma' = \mathrm{e}^{\phi'/T}$ and $\sigma_h/\sigma = \mathrm{e}^{\phi/T}$, respectively. After some calculation, it can be shown that $\varepsilon_n(\lambda) > 0$ with $n \geqslant 2$ in the $T = 0$ limit and hence equation (4.5) reduces to

$$\begin{aligned}
\varepsilon_1(\lambda) &= H - 2\pi\left[a_1(\lambda-\kappa)+a_1(\lambda+\kappa)\right] - a_2 * \varepsilon_1^-(\lambda) - a_3 * \phi'^-(\lambda) + a_1 * \phi^-(\lambda) \\
\phi'(\lambda) &= \frac{3H}{2} + A - 2\pi\left[a_2(\lambda-\kappa)+a_2(\lambda+\kappa)\right] - a_3 * \varepsilon_1^-(\lambda) \\
&\quad -a_4 * \phi'^-(\lambda) + a_2 * \phi^-(\lambda) \\
\phi(\lambda) &= -\frac{H}{2} + A + a_1 * \varepsilon_1^-(\lambda) + a_2 * \phi'^-(\lambda)
\end{aligned} \tag{5.1}$$

and the free energy (4.6) becomes

$$\begin{aligned}
\frac{G-E_0}{2N_a} &= -A - \frac{1}{2}\int \mathrm{d}\lambda\,[g_0(\lambda-\kappa)+g_0(\lambda+\kappa)]\,\varepsilon_1^+(\lambda) \\
&\quad +\frac{1}{2}\int \mathrm{d}\lambda\,[g_1(\lambda-\kappa)+g_1(\lambda+\kappa)]\left[\phi'^-(\lambda)+\phi^-(\lambda)\right]
\end{aligned} \tag{5.2}$$

where $\varepsilon_1^+(\varepsilon_1^-) = \varepsilon_1$ for $\varepsilon_1 > 0\,(\varepsilon_1 < 0)$ while $\varepsilon_1^+(\varepsilon_1^-) = 0$ for $\varepsilon_1 \leqslant 0\,(\varepsilon_1 \geqslant 0)$, with $\phi^{\pm}$ and $\phi'^{\pm}$ having the same definitions. Note that all functions are invariant under

the transformation $\lambda \to -\lambda$. In the reduced Bethe ansatz equation (5.1), κ plays a very critical role for the λ dependence of the newly introduced functions, ϵ_n, ϕ' and ϕ. Since it is difficult to deal analytically with an arbitrary value of κ, from now on we restrict ourselves to the case of $\kappa \ll \ln 2$. For the half-filled ground state without the magnetic field, $\varepsilon_1(\lambda) = -4\pi[g_0(\lambda-\kappa)+g_0(\lambda+\kappa)]$, $\phi'(\lambda) \geqslant 0$, and $\phi(\lambda) \geqslant 0$ with the constraint of the zero-T and zero-field chemical potential $A_0 \geqslant 8\pi g_1(\lambda)$. The corresponding free energy is obtained as

$$\frac{G-E_0}{2N_a} = -A_0 \tag{5.3}$$

shifted by $-A_0$ per site.

What we present here are the solutions for an almost half-filled band under the small magnetic field of $H \ll 1/\cosh(\pi\kappa/2)$ with $\phi'(\lambda) \geqslant 0$, so that one may concentrate only on two coupled equations,

$$\begin{aligned}\varepsilon_1(\lambda) &= \frac{H}{2} - 2\pi\,[g_0(\lambda-\kappa)+g_0(\lambda+\kappa)] + \int_{-Q}^{Q} \mathrm{d}\lambda'\, g_0(\lambda-\lambda')\phi(\lambda') \\ &\quad + \left(\int_{-\infty}^{-B} + \int_{B}^{\infty}\right) \mathrm{d}\lambda'\, g_1(\lambda-\lambda')\varepsilon_1(\lambda') \\ \phi(\lambda) &= \left(-\frac{H}{2}+A\right) + \int_{-B}^{B} \mathrm{d}\lambda'\, a_1(\lambda-\lambda')\varepsilon_1(\lambda')\end{aligned} \tag{5.4}$$

where B and Q are positive parameters satisfying $\varepsilon(\pm B) = 0$ and $\phi(\pm Q) = 0$, respectively. In the limit we are considering, B is always much larger than Q. Hence, for $\lambda > 0$,

$$\int_{-Q}^{Q} \mathrm{d}\lambda'\, g_0(\lambda+B-\lambda')\phi(\lambda') \simeq g_0(\lambda+B)\int_{-Q}^{Q} \mathrm{d}\lambda'\, \mathrm{e}^{\frac{1}{2}\pi\lambda'}\phi(\lambda')$$

and the first equation in (5.4) is rewritten as

$$y(\lambda) = \frac{H}{2} - R(\kappa, Q)g_0(\lambda+B) + \int_0^{\infty} \mathrm{d}\lambda' \left[g_1(\lambda-\lambda') + g_1(\lambda+\lambda'+2B)\right] y(\lambda') \tag{5.5}$$

where $y(\lambda) = \varepsilon_1(\lambda+B)$ and

$$R(\kappa, Q) = 4\pi\cosh\frac{\pi}{2}\kappa - \int_{-Q}^{Q} \mathrm{d}\lambda'\, \mathrm{e}^{\frac{1}{2}\pi\lambda'}\phi(\lambda'). \tag{5.6}$$

Equation (5.5) is similar to the Heisenberg-spin model except for $R(\kappa, Q)$ which contains information related to the coupling constant κ and the chemical potential. Replacing $y(\lambda)$ with $y_1(\lambda) + y_2(\lambda)$, we have

$$\begin{aligned} y_1(\lambda) &= \frac{H}{2} - R(\kappa, Q)g_0(\lambda+B) + \int_{-\infty}^{\infty} \mathrm{d}\lambda'\, g_1(\lambda-\lambda')y_1^+(\lambda') \\ y_2(\lambda) &= \int_{-\infty}^{\infty} \mathrm{d}\lambda'\, g_1(\lambda+\lambda'+2B)y_1^+(\lambda') + \int_{-\infty}^{\infty} \mathrm{d}\lambda'\, g_1(\lambda-\lambda')y_2^+(\lambda') \end{aligned} \tag{5.7}$$

where $y_i^+(\lambda) = y_i(\lambda)$ if $\lambda > 0$ and $y_i^+ = 0$ otherwise. Using the Wiener–Hopf method [25], it is obtained that

$$\tilde{y}^+(w) \simeq \left[\pi H G^-(0)\delta^+(w) - \pi\,\mathrm{e}^{-\frac{1}{2}\pi B} G^-\left(-\mathrm{i}\frac{\pi}{2}\right) R(\kappa, Q)\delta^+\left(w+\mathrm{i}\frac{\pi}{2}\right)\right] G^+(w) \tag{5.8}$$

where

$$G^{\pm}(w) \equiv \frac{\sqrt{2\pi}\, f_{\pm}(w)}{\Gamma\left(\frac{1}{2} \mp \mathrm{i}w/\pi\right)}$$

and

$$f_{\pm}(w) = \left[\frac{\mp \mathrm{i}w + 0}{\mathrm{e}\pi}\right]^{\mp \mathrm{i}w/\pi}.$$

Applying the boundary condition $y^{+}(0) = 0$, we have

$$B = -\frac{2}{\pi} \ln\left[\frac{HG^{-}(0)}{G^{+}(\mathrm{i}\pi/2)R(\kappa, Q)}\right]$$

and

$$\tilde{y}^{+}(w) \simeq \pi G^{-}(0) H \left[\delta^{+}(w) - \delta^{+}\left(w + \mathrm{i}\frac{\pi}{2}\right)\right] G^{+}(w). \tag{5.9}$$

Meanwhile, the second equation in (5.4) can be rewritten in the more convenient form

$$\begin{aligned}\phi(\lambda) = A - 2\pi\left[g_1(\lambda - \kappa) + g_1(\lambda + \kappa)\right] + \int_{-Q}^{Q} \mathrm{d}\lambda'\, g_1(\lambda - \lambda')\phi(\lambda') \\ - \int_{0}^{\infty} \mathrm{d}\lambda' \left[g_0(\lambda - \lambda' - B) + g_0(\lambda + \lambda' + B)\right] y^{+}(\lambda').\end{aligned} \tag{5.10}$$

Substituting equation (5.9) into the above and using the relation

$$\begin{aligned}g_1(\lambda) &= \frac{1}{4\pi}\mathrm{Re}\left[\psi\left(1 + \mathrm{i}\frac{\lambda}{4}\right) - \psi\left(\frac{1}{2} + \mathrm{i}\frac{\lambda}{4}\right)\right] \\ &\simeq \frac{1}{2\pi}\left[\ln 2 - \frac{3}{16}\zeta(3)\lambda^2 + \frac{15}{256}\zeta(5)\lambda^4 + \cdots\right] \qquad \lambda \ll 1\end{aligned}$$

where ψ is the digamma function and ζ is the Riemann zeta function, $\phi(\lambda)$, which satisfies, for $|\lambda| \leqslant Q$,

$$\begin{aligned}\phi(\lambda) = A - 2\pi\left[g_1(\lambda - \kappa) + g_1(\lambda + \kappa)\right] + \frac{1}{2\pi}\left[\ln 2 - \frac{3}{16}\zeta(3)\lambda^2\right] R_1 \\ + \frac{3}{16\pi}\zeta(3)\lambda R_2 - \frac{3}{32\pi}\zeta(3) R_3 - \frac{\cosh\frac{\pi}{2}\lambda}{\pi R(\kappa, Q)} H^2\end{aligned} \tag{5.11}$$

with

$$R_n \equiv \int_{-Q}^{Q} \mathrm{d}\lambda' \left(\lambda'\right)^{n-1} \phi(\lambda').$$

Hence $R(\kappa, Q)$ in (5.6) can be expanded as

$$R(\kappa, Q) = 4\pi \cosh\frac{\pi}{2}\kappa - \left[R_1 + \frac{\pi}{2}R_2 + \frac{\pi^2}{8}R_3 + \cdots\right]$$

with R_n having the leading order of Q^{n+2}. Let us suppose that κ is close to Q so that both have the same order of magnitude. Then the boundary condition $\phi(Q) = 0$ yields the relation of the chemical potential and $\phi(\lambda)$ for $|\lambda| \leqslant Q$ with Q and H

$$\begin{aligned}\phi(\lambda) &\simeq -\left[\frac{3}{8}\zeta(3) - \frac{H^2}{32}\left(1 - \frac{\pi^2}{8}\kappa^2\right)\right]\left(Q^2 - \lambda^2\right) \qquad |\lambda| \leqslant Q \\ A &\simeq 2\ln 2 - \frac{3}{8}\zeta(3)\kappa^2 + \frac{H^2}{4\pi^2}\left(1 - \frac{\pi^2}{8}\kappa^2\right) - \left[\frac{3}{8}\zeta(3) - \frac{H^2}{32}\left(1 - \frac{\pi^2}{8}\kappa^2\right)\right] Q^2\end{aligned} \tag{5.12}$$

where

$$R(\kappa, Q) \simeq 4\pi \cosh \frac{\pi}{2}\kappa - \frac{1}{2}\zeta(3)Q^3$$

has been used. Note that H and Q are also related to B as

$$B \simeq -\frac{4}{\pi} \ln \left[\frac{\sqrt{2}H}{8\pi^2 \cosh \frac{\pi}{2}\kappa - \pi\zeta(3)Q^3} \right].$$

The resulting free energy becomes

$$\frac{G - E_0}{2N_a} = -2\ln 2 + \frac{3}{8}\zeta(3)\left(\kappa^2 + Q^2\right) - \left[\frac{1}{2\pi^2} + \frac{1}{32}\left(Q^2 - \kappa^2\right)\right] H^2 \quad (5.13)$$

up to order Q^2 or κ^2, from which one can straightforwardly obtain the corresponding magnetic susceptibility,

$$\chi_s = \frac{1}{\pi^2} + \frac{1}{16}\left(Q^2 - \kappa^2\right). \quad (5.14)$$

In the limit of Q and κ being zero, it leads us to the susceptibility of the two-chain Heisenberg spin-$\frac{1}{2}$ model in a small magnetic field [10].

6. Concluding remarks

An integrable coupled supersymmetric t–J model was considered as an extension of the coupled Heisenberg spin chains. Our Hamiltonian reduces to a single t–J chain with $2N_a$ sites in the limit of $\kappa \to \infty$, whereas we have two independent t–J chains when $\kappa = 0$. The corresponding Bethe ansatz equations and energy were constructed through the diagonalization of the extended transfer matrix within the framework of BFF grading. The thermodynamic Bethe ansatz equations were also obtained. At zero temperature we found the energy and the magnetic susceptibility under a weak magnetic field, limiting ourselves to the case of $\kappa \ll \ln 2$. This limit led us to the usual Wiener–Hopf-type integral equations so that the ground state was continuously changed as the magnetic field increased. Hence the spin excitations show no gap at least with a weak coupling between the t–J chains. This might be a consequence of the unnatural terms describing three-neighbouring site interactions in Hamiltonian (2.9) and the limit of small κ. We cannot neglect the three-neighbouring site interactions because of the integrability of the model [26], while the large-coupling limit can be treated numerically and is quite interesting in the sense that it is expected to provide some new features. Detailed work will be published elsewhere.

It is also very interesting to study the low-temperature properties such as the specific heat and the magnetic susceptibility at a finite temperature. These properties can be obtained by solving equations (4.5) and (4.6) and will also be published elsewhere.

Acknowledgments

This work is supported partially by the Brain Korea 21 Project through the Ministry of Education (Republic of Korea) and partially by Korea Research Foundation Grant (KRF-2000-015-DP0151).

References

[1] Uehara M, Nagata T, Akimitsu J, Takahashi H, Mori N and Kinoshita K 1996 *J. Phys. Soc. Japan* **65** 2764
[2] Curely J, Georges R, Drillon M, Belaiche M and Benhoujia K 1992 *Phys. Rev.* B **46** 3527
[3] Hiroi Z, Azuma M, Takano M and Bando Y 1991 *J. Solid State Chem.* **95** 230
[4] Takano M, Hiroi Z, Azuma M and Takede Y 1991 *Japan. J. Appl. Phys.* **7** 3
[5] For a review, see Dagotto E and Rice T M 1996 *Science* **271** 618
[6] Haldane F D M 1982 *Phys. Rev.* B **25** 4925
Haldane F D M 1982 *Phys. Rev.* B **26** 5257 (erratum)
[7] Nomura K and Okamoto K 1994 *J. Phys. A: Math. Gen.* **27** 5773
[8] Allen D and Sénéchal D 1997 *Phys. Rev.* B **55** 299
[9] Frahm H and Rödenbeck C 1996 *Europhys. Lett.* **33** 47
[10] Popkov V Y and Zvyagin A A 1993 *Phys. Lett.* A **175** 295
[11] Wang Y 1999 *Phys. Rev.* B **60** 9236
[12] Batchelor M T and Maslen M 1999 *J. Phys. A: Math. Gen.* **32** L377
[13] Links J and Foerster A 2000 *Phys. Rev.* B **62** 65
[14] Anderson P W 1987 *Science* **235** 1196
[15] Zhang F C and Rice T M 1988 *Phys. Rev.* B **37** 3759
[16] Sutherland B 1975 *Phys. Rev.* B **12** 3795
[17] Schlottmann P 1987 *Phys. Rev.* B **36** 5177
[18] Essler F H L and Korepin V E 1992 *Phys. Rev.* B **46** 9147
[19] Park S and Lee K 1998 *J. Phys. A: Math. Gen.* **31** 6569
[20] des Cloizeaux J and Gaudin M 1966 *J. Math. Phys.* **7** 1384
[21] Korepin V E, Bogoliubov N M and Izergin A 1993 *Quantum Inverse Scattering Method and Correlation Functions* (New York: Cambridge University Press) p 97
[22] des Cloizeaux J and Pearson J J 1962 *Phys. Rev.* **128** 2131
[23] Yang C N and Yang C P 1969 *J. Math. Phys.* **10** 1115
[24] Takahashi M 1971 *Prog. Theor. Phys.* **46** 401
[25] Babujian H M 1983 *Nucl. Phys.* B **215** 317
[26] Zvyagin A A 1995 *Phys. Rev.* B **52** 15050